Differential model

Lotka–Volterra equations

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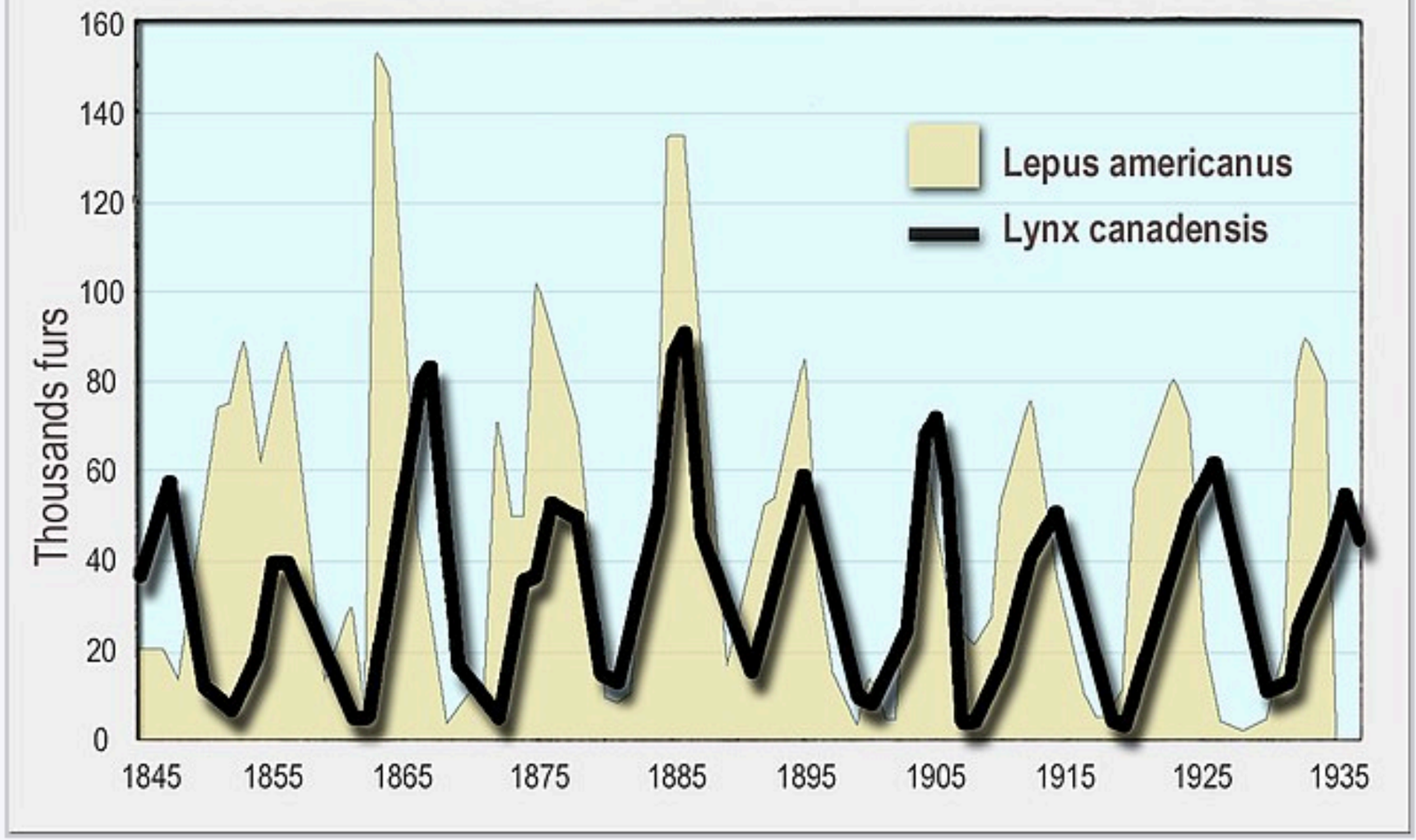
This article is about the predator-prey equations. For the competition equations, see Competitive Lotka–Volterra equations.

The **Lotka–Volterra equations**, also known as the **Lotka–Volterra predator–prey model**, are a pair of first-order nonlinear differential equations, frequently used to describe the dynamics of biological systems in which two species interact, one as a predator and the other as prey. The populations change through time according to the pair of equations:

$$\frac{dx}{dt} = \alpha x - \beta xy, \qquad \rho_r$$
$$\frac{dy}{dt} = -\gamma y + \delta xy,$$

where

- the variable x is the population density of prey (for example, the number of rabbits per square kilometre);
- the variable y is the population density of some predator (for example, the number of foxes per square kilometre);
- $\frac{dy}{dt}$ and $\frac{dx}{dt}$ represent the instantaneous growth rates of the two populations;
- t represents time;
- The prey's parameters, α and β, describe, respectively, the maximum prey per capita growth rate, and the effect of the presence of predators on the prey death rate.
- The predator's parameters, γ, δ, respectively describe the predator's per capita death rate, and the effect of the presence of prey on the predator's growth rate.
- All parameters are positive and real.



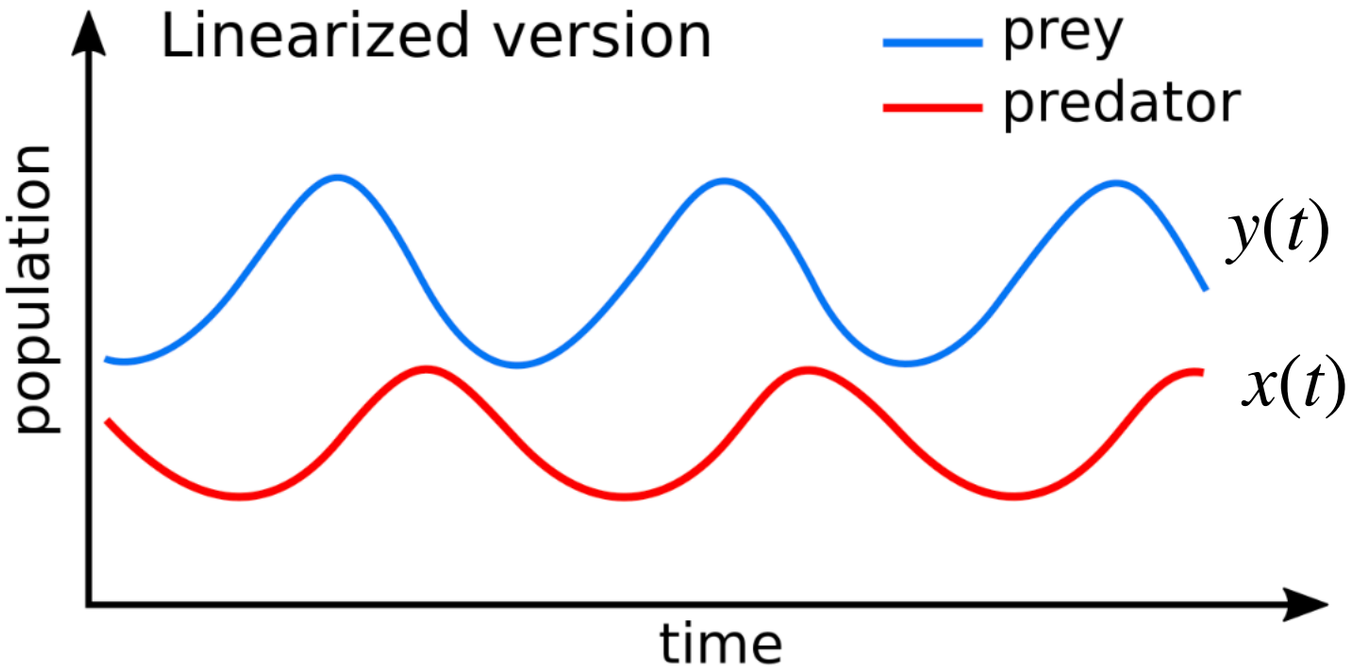
Numbers of snowshoe hare (yellow, background) and Canada lynx (black line, foreground) furs sold to the Hudson's Bay Company. Canada lynxes eat snowshoe hares.

Solutions to the equations [edit]

The equations have periodic solutions. These solutions do not have a simple expression in terms of the usual trigonometric functions, although they are quite tractable.[27][28][29]

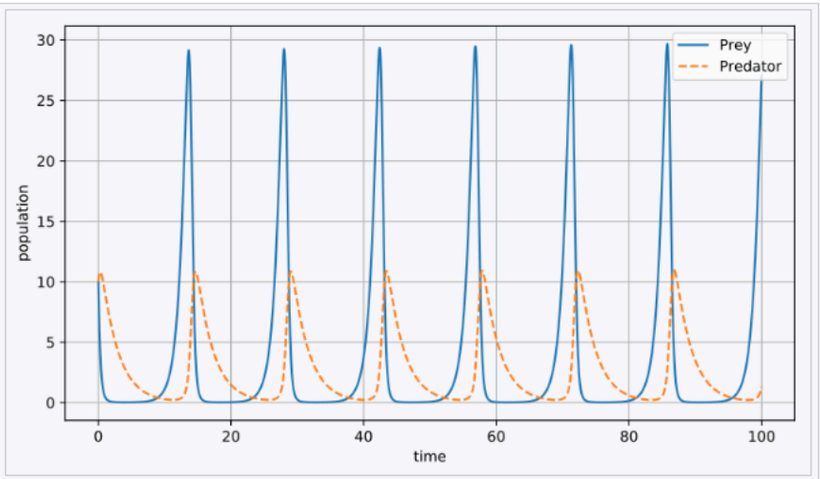
If none of the non-negative parameters α, β, γ, δ vanishes, three can be absorbed into the normalization of variables to leave only one parameter: since the first equation is homogeneous in x, and the second one in y, the parameters β/α and δ/γ are absorbable in the normalizations of y and x respectively, and γ into the normalization of t, so that only α/γ remains arbitrary. It is the only parameter affecting the nature of the solutions.

A linearization of the equations yields a solution similar to simple harmonic motion[30] with the population of predators trailing that of prey by 90° in the cycle.



A simple example [edit]

Suppose there are two species of animals, a rabbit (prey) and a fox (predator). If the initial densities are 10 rabbits and 10 foxes per square kilometre, one can plot the progression of the two species over time; given the parameters that the growth and death rates of rabbits are 1.1 and 0.4 while that of foxes are 0.1 and 0.4 respectively. The choice of time interval is arbitrary.



Population dynamics for rabbit and fox problem mentioned aside.

One may also plot solutions parametrically as orbits in phase space, without representing time, but with one axis representing the number of prey and the other axis representing the densities of predators for all times.

This corresponds to eliminating time from the two differential equations above to produce a single differential equation

$$\frac{dy}{dx} = -\frac{y}{x}\frac{\delta x - \gamma}{\beta y - \alpha}$$

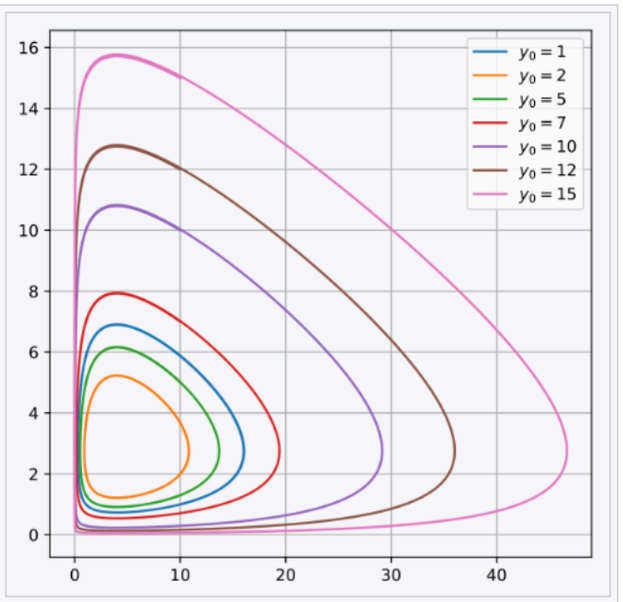
relating the variables *x* (predator) and *y* (prey). The solutions of this equation are closed curves. It is amenable to separation of variables: integrating

$$\frac{\beta y - \alpha}{y}\,dy + \frac{\delta x - \gamma}{x}\,dx = 0$$

yields the implicit relationship

$$V = \delta x - \gamma \ln(x) + \beta y - \alpha \ln(y),$$

where *V* is a constant quantity depending on the initial conditions and conserved on each curve.



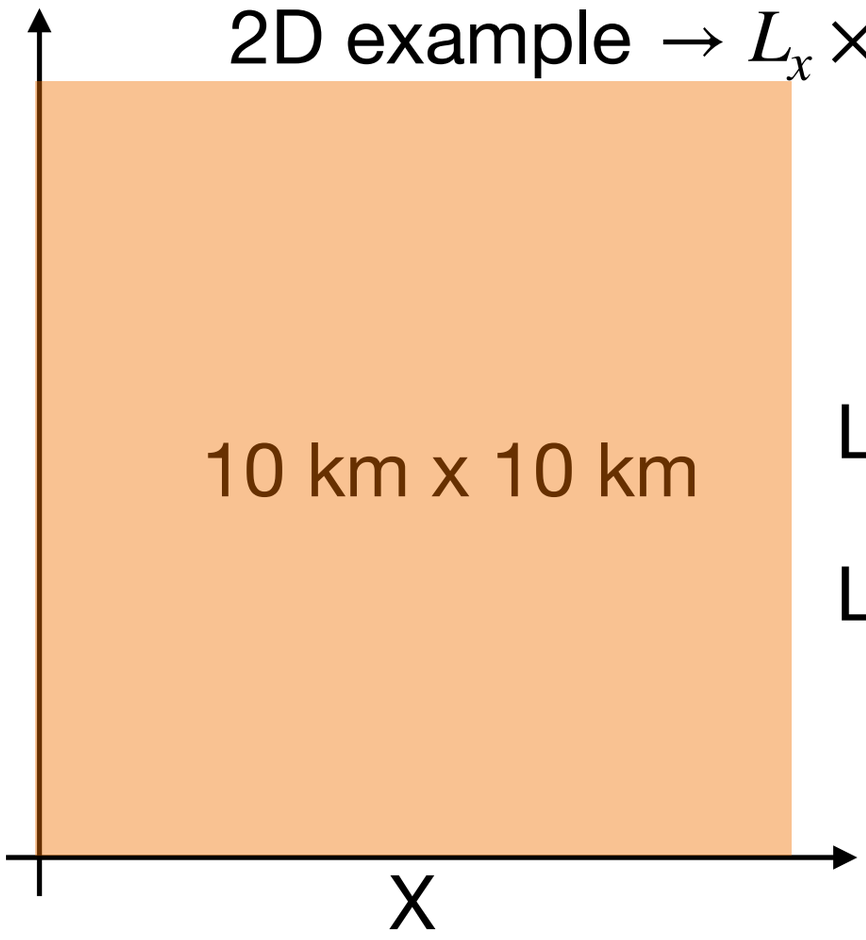
Phase-space plot for the predator prey problem for various initial conditions of the predator population.

An aside: These graphs illustrate a serious potential limitation in the application as a biological model: for this specific choice of parameters, in each cycle, the rabbit population is reduced to extremely low numbers, yet recovers (while the fox population remains sizeable at the lowest rabbit density). In real-life situations, however, chance fluctuations of the discrete numbers of individuals might cause the rabbits to actually go extinct, and, by consequence, the foxes as well. This modelling problem has been called the "atto-fox problem", an atto-fox being a notional 10^{-18} of a fox.[31][32] A density of 10^{-18} foxes per square kilometre equates to an average of approximately 5×10^{-10} foxes on the surface of the earth, which in practical terms means that foxes are extinct.

Network model

Simulation domain

2D example $\rightarrow L_x \times L_y$

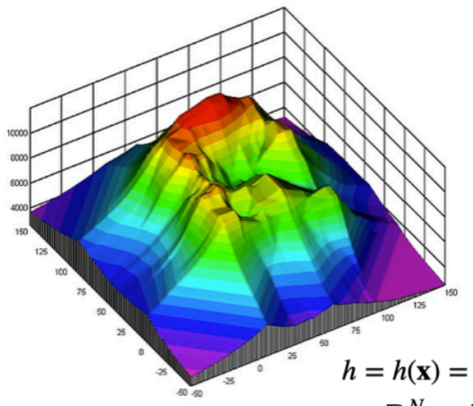


Lx=100

Ly=100

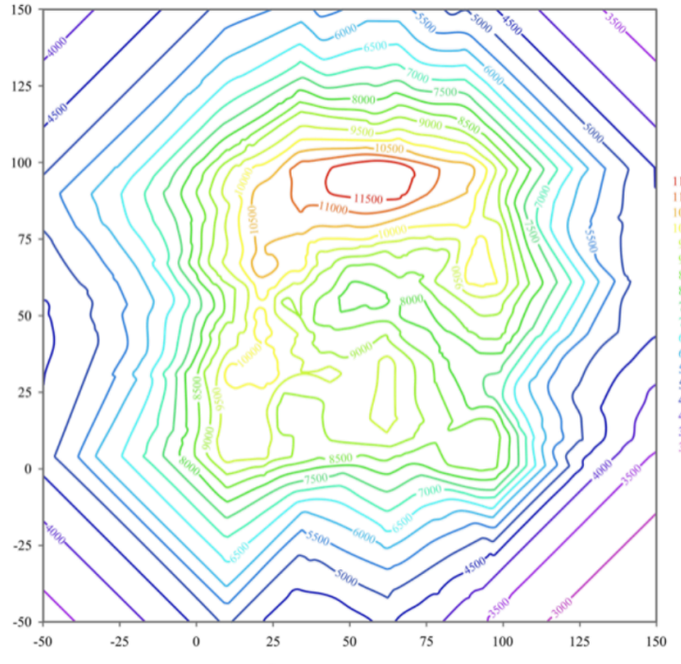
3D example $\rightarrow L_x \times L_y \times L_z$





$h = h(\mathbf{x}) = h(x, y)$

$\mathbb{R}^N \rightarrow \mathbb{R}^1$

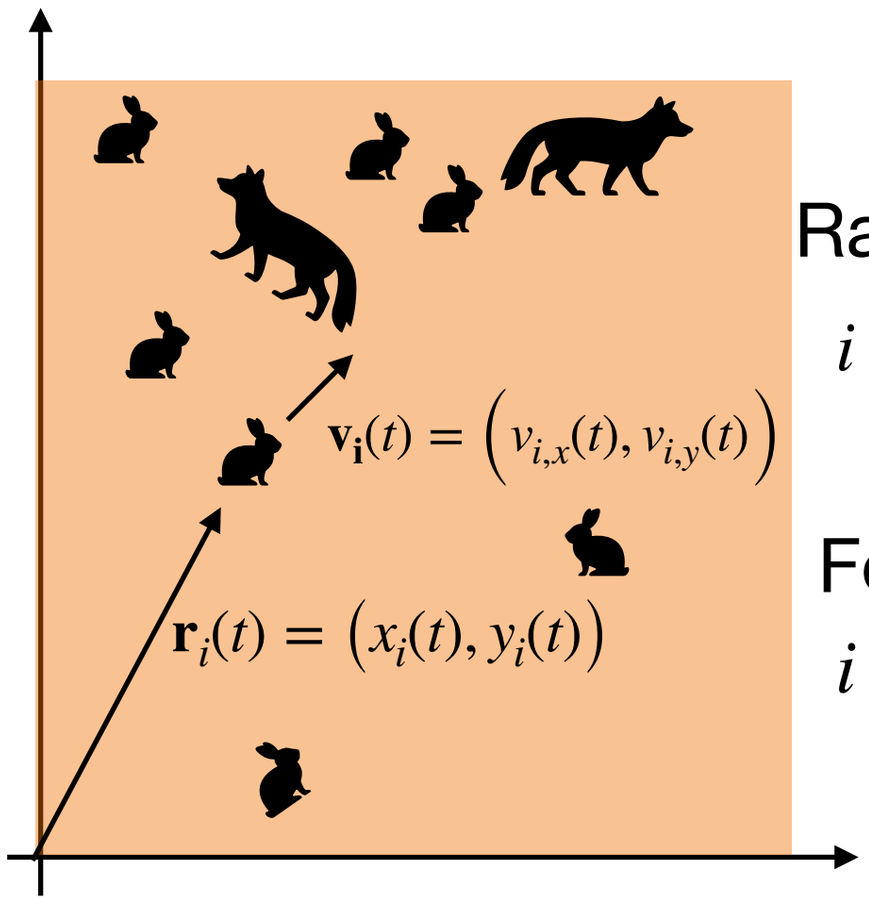


Contour plot

Fix h at constant intervals and let x and y vary

Initialization

Initialization: outside of time loop



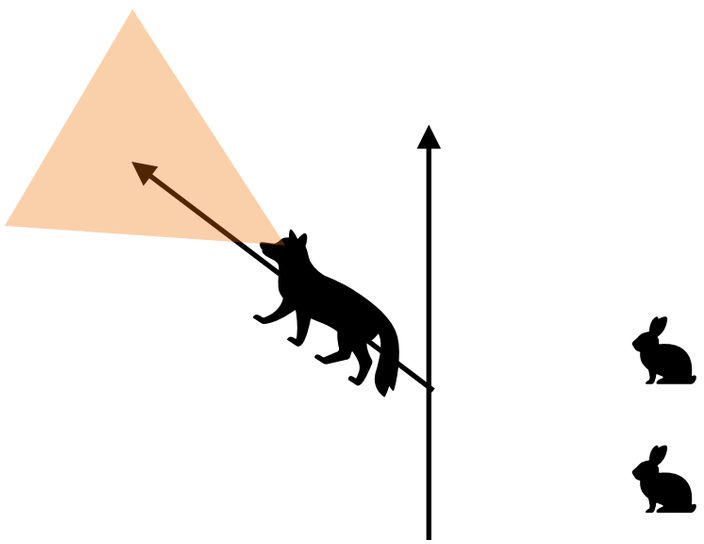
Rabbit node attributes

$$i = ID, T = R, x_i(t), y_i(t), v_x(t), v_y(t), Age, Sex, angle$$

Fox node attributes

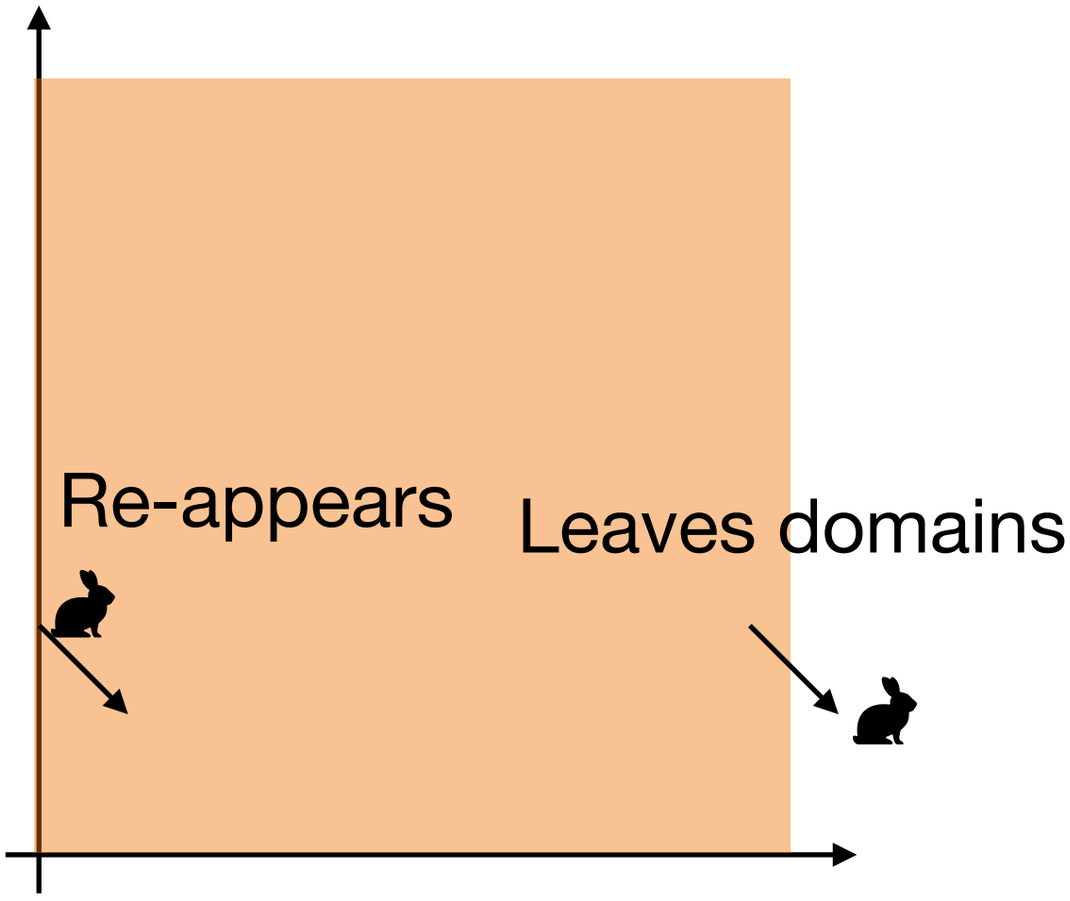
$$i = 2, T = F, x(t), y(t), v_x(t), v_y(t), Age, Sex, angle$$

Possible increase in model complexity

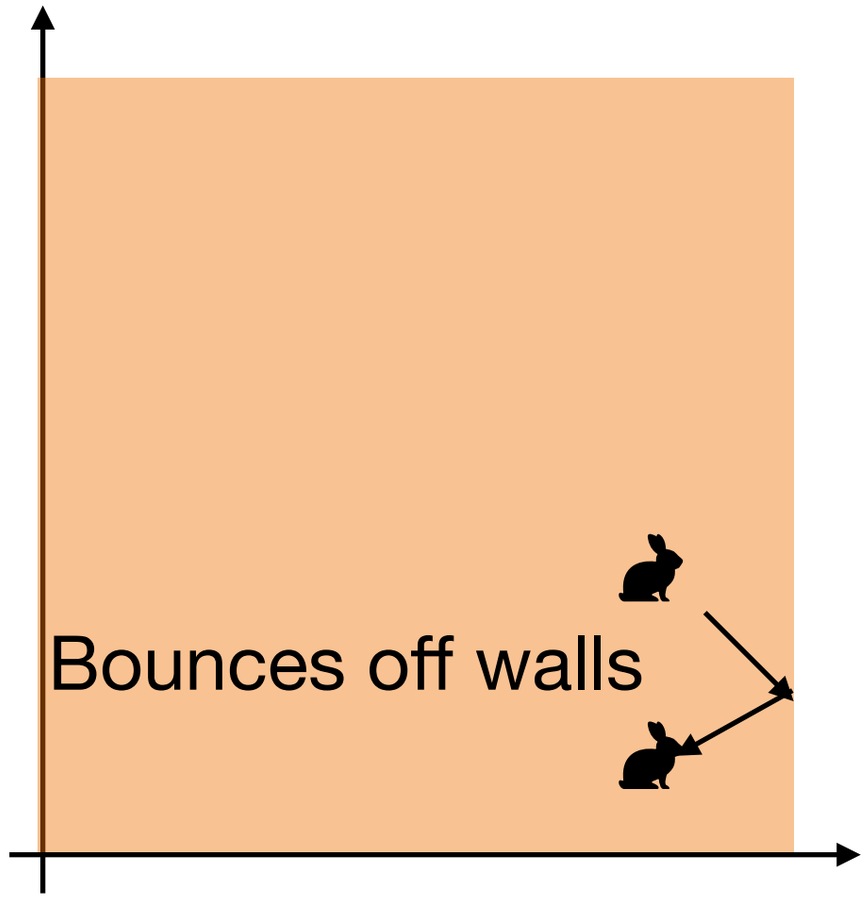


Boundary conditions

Periodic boundary conditions

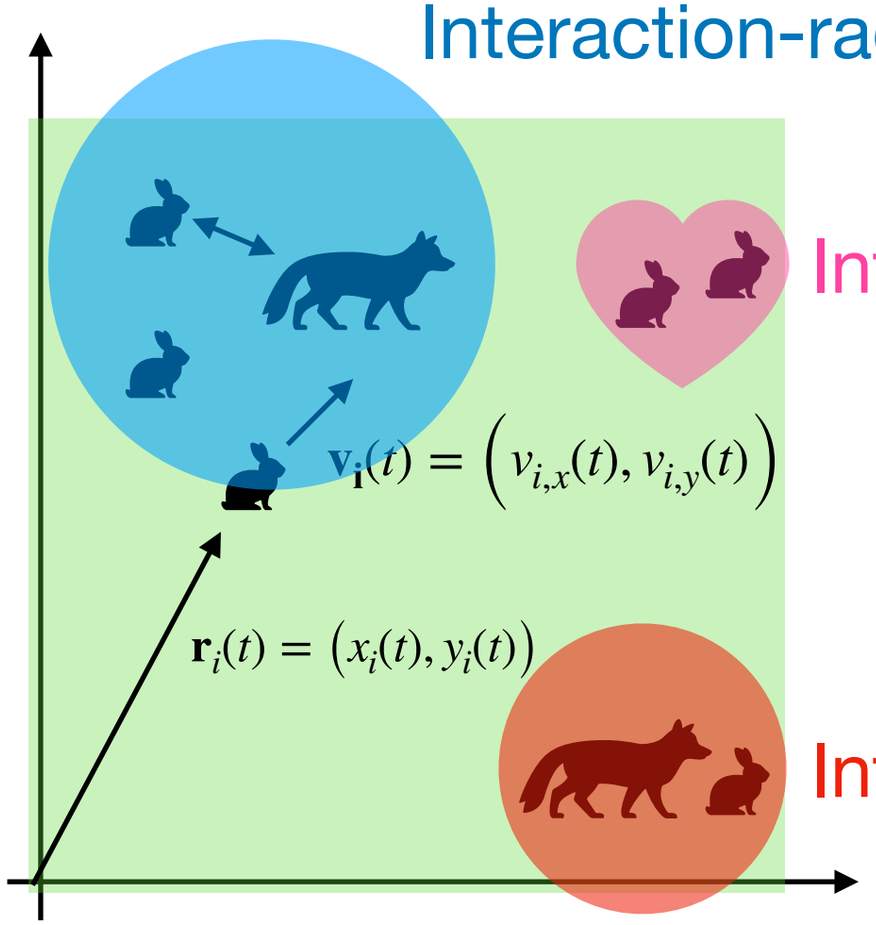


Fixed boundary conditions



Interactions

Interaction-radius form temporally variable bonding (edge creation)



Interaction-radius-1 (visual range)

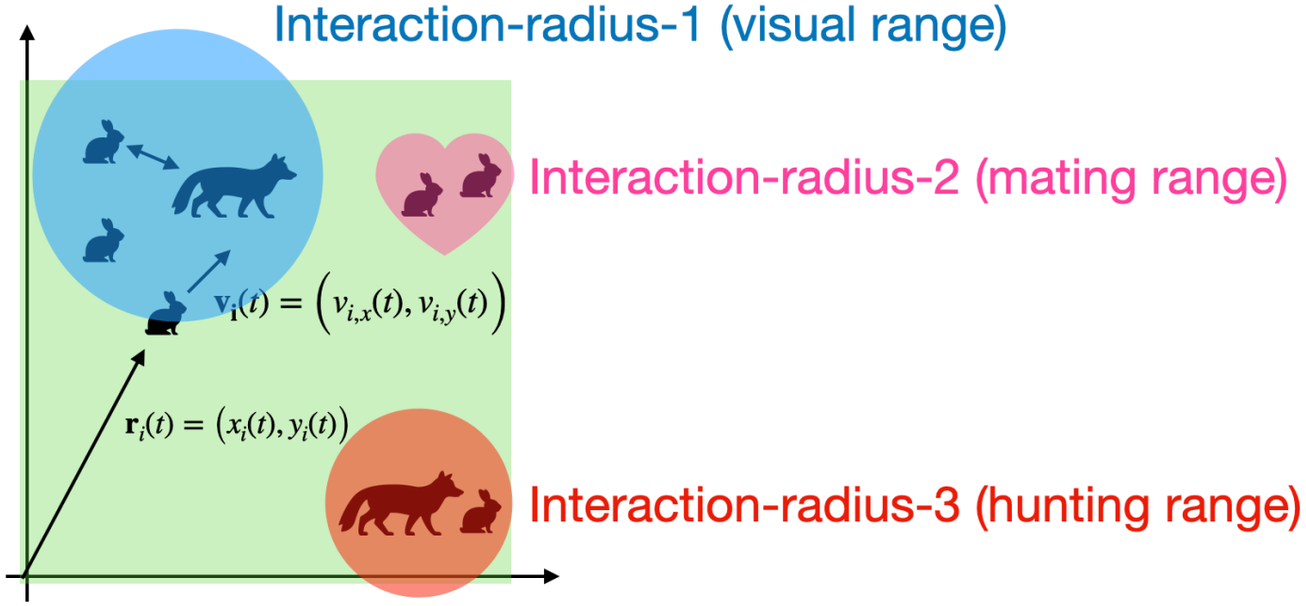
Interaction-radius-2 (mating range)

Interaction-radius-3 (hunting range)

Run simulations

Time loop

dt=1 = day



Pseudo-code

```
For indx in range(0, 365)
    t=indx*dt
    Compute nodal distance()
        d_ij
    Node_update()
        Increase age()
        position()
            x=x+dx
            y=y+dx
        give_birth()
        die_of_old_age()
    Form neighbor list(d_ij)
    Interaction_rules()
    R —> R
        mate()
        fight()
    R —> F
        Flee()
    F —> R
        chase()
        eat()
    F —> F
        mate()
```